Nested intervals Thm $\Rightarrow$ Intermediate value Thm

$\Updownarrow$

Bolzano Weierstrass Thm $\Leftrightarrow$ Compact set

limsup, liminf

limsup = greatest subsequential limit

liminf = least subsequential limit

Subsequence $\to$ (☆$_1$)

Sequence

L is the limit of seq (x_n) if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon$ $\forall n \geq N$

Examples for calculation :

$\lim_{n\to\infty} \frac{n+(-1)^n}{n^2-1} = 0$

$\lim_{n\to\infty} \frac{n^2+n}{2n^2-1} = \frac{1}{2}$

See 3.1.6 Examples

Some other techniques :

- root test
- ratio test
- Let $(a_n) \subset \mathbb{R}_{>0}$

if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$, then $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

e.g. $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = 0$

Divergence = does not converge in $\mathbb{R}$

e.g.

1. diverge to ∞ : $\forall M > 0$, $\exists N \in \mathbb{N}$ s.t. $x_n > M$ $\forall n \geq N$
2. diverge to $-\infty$: $\forall M > 0$, $\exists N \in \mathbb{N}$ s.t. $x_n < -M$ $\forall n \geq N$
3. $(x_n) = (-1)^n$

- A sequence converges iff it is Cauchy eg.
  - Contractive seq

    e.g. Ex for § 3.5 Q11 [if $y_1 < y_2$ and $y_n := \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2}$. Show that it converges and find its limit.]

  - $\sum_{n=1}^{\infty} \frac{1}{n}$ does not exist

☆☆ [You cannot pass limit on both sides to find its limit] ☆☆

- Determining whether a sequence converges
  - Monotone convergence Thm

    eg. • $\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists. , • $\sum_{n=1}^{\infty}\frac{1}{n^2}$ exists

    • $\begin{cases} x_1 = 1 \\ x_{n+1} = \frac{1+x_n}{2+x_n} \end{cases}$ and find its limit

    • Q6 in Ex for § 3.3. Check that it is monotone, whether it is increasing or decreasing depends on x_1, c. (Easier version is Q4)
  - Cauchy criteria

Some special seq and some comparison :

- for $0 < r < 1$, $\lim_{n\to\infty} r^n = 0$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} \sqrt[n]{c} = 1$ $\forall c > 0$
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

See HW III A in Class B. Q3

(☆$_1$) : Let $(x_n) \subset \mathbb{R}$, $L \in \mathbb{R}$

$\lim_{n\to\infty} x_n = L$ iff every subseq (x_{n_k}) of (x_n) admits a subseq $(x_{n_{k_j}})$ s.t. $\lim_{j\to\infty} x_{n_{k_j}} = L$

Definition of "cluster point of A"

Limit of a fcn

- As $x \to c^-$
- As $x \to c^+$
- As $x \to \infty$
- As $x \to -\infty$

See class B Tutorial Notes p.20.

Examples: See 2050B (2015-2016) HW VI Q4.

As $x \to c$

Examples: Show that (i) $\lim_{x\to 3} \frac{x+1}{x-2} = 4$

(ii) $\lim_{x\to 3} \frac{x^3+1}{x-2} = 28$

(Read Class B Lecture Notes titled "examples")

- For more examples in taking limit, see p. 116 in our textbook

$\lim_{x\to c} f(x)$ is independent of $f(c)$

$\Rightarrow$ • Suppose $\lim_{x\to c} f(x) = l$, $\lim_{y\to l} g(y) = L$, we cannot conclude that $\lim_{x\to c} g(f(x)) = L$, because f may be constantly l.

Sequential criteria ($\star_2$)

- $\lim_{x\to 0} \sin(1/x^2)$ does not exist
- Q5, 6 in Tuto 6
- proof of Max-Min Thm

(Read Class B Lecture NOTES titled "cts functions on interval")

OR Proposition 3.5 in our lecture notes

Continuous fcn

- Max-Min Thm: Let $f: A \to \mathbb{R}$ be continuous, where A is compact or a closed and bdd interval. Then, $\exists\, x_1, x_2 \in A$ s.t. $f(x_1) \le f(x) \le f(x_2)$ $\forall\, x \in A$
- Intermediate Value Thm [Theorem 5.1 in lecture NOTES]

uniform continuous

Examples:

- Tutorial 7 (Last few lines) [Use Sequential criteria to check that f is not uniformly cont]
- Tutorial 8, 9

[NOTE $\lim_{x\to 0} \frac{\sin x}{x} = 1$]

- 5.4.8 Continuous Ext^n Thm (Textbook): proof: ① If f is uniformly cont, then f sends Cauchy seq to Cauchy seq. Combine ① a ($\star_2$). ✗

($\star_2$): Suppose $f: A \to \mathbb{R}$ and c is a cluster pt of A.

If $\lim_{n\to\infty} f(x_n)$ exists in $\mathbb{R}$ for any sequence (x_n), where (i) $(x_n) \subset A \setminus \{c\}$, (ii) $\lim_{n\to\infty} x_n = c$,

then $\lim_{x\to c} f(x)$ exists

- The point is that we don't need to propose an $L \in \mathbb{R}$ st. $\lim_{n\to\infty} f(x_n) = L$ for any seq (x_n) ...

- $\nexists\, f: \mathbb{R} \to \mathbb{R}$ cont precisely on $\mathbb{Q}$ (NOT for your revision)